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ADP010479

TITLE: Unified Flight Mechanics and Aeroelasticity for Accelerating, Maneuvering, Flexible Aircraft

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ADP010474 thru ADP010498

Unified "
Flight Mechanics and Aeroelasticity
for
Accelerating, Maneuvering, Flexible Aircraft

Dr. James J. Olsen
Olsen Engineering Consulting
1186 Tralee Trail
Dayton, Ohio 45430-1219
United States

Abstract

This paper reveals new insights in the aeroelasticity and flight mechanics of flexible aircraft by obtaining and solving the equations of motion for a flexible, accelerating, rotating aircraft. We illustrate the approach for three cases of increasing complexity: The first case is a "sprung" pendulum. It shows when rigid body angular velocities can be important in the flexibility equations as they approach as the flexible frequencies. The second case is a typical section airfoil on an accelerating, rotating fuselage. It applies Lagrange's equations to a longitudinal problem in inertial coordinates, then transforms the equations to noninertial, body - fixed coordinates for solution. It also shows when rigid body rotations and longitudinal accelerations must be included in the flexibility equations. The third case is the general longitudinal/lateral motion of an accelerating, rotating, flexible vehicle. Rather than setting up the general problem in inertial coordinates and then transforming to body - fixed coordinates, instead we use the idea of "quasi - coordinates". We establish a general form for Lagrange's equations in the noninertial, body - fixed coordinates. The paper gives the general equations and reduces them to a special case of a "flat" airplane. It also gives guidelines as to when the rigid body rotations and accelerations are important factors in the flexibility equations.

1. Introduction

For many years there has been a search for a practical set of "unified" equations of motion that can be used in all of the disciplines of aerodynamics, structures and stability and control of flexible aircraft. Such an approach would allow the customary determination of the effects of structural flexibility on aircraft performance, stability and air loads. An added benefit is that it would also allow us to determine the effects of the "rigid body" motions on aeroelastic characteristics such as control - effectiveness, divergence and flutter. Further (and most importantly), it would allow all of those engineering problems to be treated by subsets of a single set of "unified" equations. In effect we want to convert the aeroelastic problems into coordinate systems and equations that are conventional for aircraft flight mechanics, stability and control.

In many aircraft applications, the mutual coupling of rigid body and flexible motions has been small because the vehicle angular velocities and flexible frequencies were well separated. However, there have been recent examples of large aircraft where flexible frequencies (say 2 Hz) begin to approach the rigid body angular velocities (say 1 Hz). Other cases have been known where the aerodynamic forces can drive the structural frequencies and the rigid body frequencies close together. In both cases the coupling effects should be accounted for in the lowest order equations of motion to obtain the correct modeling.

To develop the necessary equations we must account for the fact that the aircraft's body - fixed coordinate system is not (in general) an inertial system. Dusto et al [1], Bekir et al [2] and Waszak and Schmidt [3] are a few examples of earlier attempts which have had to leave out crucial terms or were difficult to implement. This paper shows that a practical set of equations for general problems is available through the use of energy methods, Lagrange's equations and "quasi - coordinates".

We illustrate the approach for three cases of increasing complexity: The first case is a "sprung" pendulum. It shows when rigid body angular velocities can be important in the flexibility equations as they approach as the flexible frequencies. The second case is a typical section airfoil on an accelerating, rotating fuselage. It applies Lagrange's equations to a longitudinal problem in inertial coordinates, then transforms the equations to noninertial, body - fixed coordinates for solution. It also shows when rigid body rotations and longitudinal accelerations must be included in the flexibility equations. The third case is the general longitudinal/lateral motion of an accelerating, rotating, flexible vehicle. Rather than setting up the general problem in inertial coordinates and then transforming to body - fixed coordinates, instead we use the idea of "quasi - coordinates". We establish a general form for Lagrange's equations in the noninertial, body - fixed coordinates. The paper gives the general equations and reduces them to a special case of a "flat" airplane. It also gives guidelines as to when the rigid body rotations and accelerations are important factors in the flexibility equations.

The equations become somewhat more complicated, and it is useful to examine them in three stages. First, some

Paper presented at the RTO AVT Specialists' Meeting on "Structural Aspects of Flexible Aircraft Control", held in Ottawa, Canada, 18-20 October 1999, and published in RTO MP-36.

insight is available via a simplification which considers the rigid body motions merely as constant parameters. There the rigid body motions alter the flexible frequencies of vibration, thereby altering aeroelastic stability. Second, a more exact approach is to recognize that the flexibility equations have some of the characteristics of Mathieu's classical ordinary differential equation. The similarity to Mathieu's equation introduces the possibility that the coupled rigid - body/flexible motions can be unstable within narrow ranges of frequencies and amplitudes, even without aerodynamic forces. Third, the ultimate procedure is always available - the simultaneous solution (perhaps numerically) of the fully coupled, non - linear, rigid body and flexibility equations of motion in body - fixed coordinates.

2. Lagrange's Equations

If the inertial coordinates of a dynamic system can be represented in terms of N independent generalized coordinates:

$$X = X(q_i) \quad Y = Y(q_i) \quad Z = Z(q_i)$$

Lagrange's equations [4] can describe the motion of the system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

where:

$$L = \text{Lagrangian, } T - U$$

$$Q_i = \text{Generalized Force}$$

$$T = \text{Kinetic Energy}$$

$$U = \text{Potential Energy}$$

For simple geometries, it usually is a straightforward matter to write down the inertial coordinates, inertial velocities, kinetic and potential energies, the Lagrangian, and the various derivatives. For complicated geometries, the process can become tedious, but Whittaker[5] showed that, if the kinetic energy can be expressed in terms of the coefficients m_{ij}

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j1}^{n} m_{ij}\dot{q}_i\dot{q}_j$$

Then the equations of motion can be written

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\begin{bmatrix} j\ k \\ i \end{bmatrix}\dot{q}_j\dot{q}_k = Q_i + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right) - \frac{\partial U}{\partial q_i}$$

where the Christoffel symbol is:

$$\begin{bmatrix} j\ k \\ i \end{bmatrix} = \frac{1}{2}\left(\frac{\partial m_{ji}}{\partial q_k} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i}\right)$$

Olsen[6] showed a related (and sometimes) simpler approach, noting that if we could write the partial derivatives:

$$X_i = \frac{\partial X}{\partial q_i} \quad Y_i = \frac{\partial Y}{\partial q_i} \quad Z_i = \frac{\partial Z}{\partial q_i}$$

$$X_{ij} = \frac{\partial^2 X}{\partial q_i \partial q_j} \quad Y_{ij} = \frac{\partial^2 Y}{\partial q_i \partial q_j} \quad Z_{ij} = \frac{\partial^2 Z}{\partial q_i \partial q_j}$$

We don't need the often tedious expressions for the kinetic energy, and the equations of motion take the form:

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} m_{ijk}\dot{q}_j\dot{q}_k = Q_i + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right) - \frac{\partial U}{\partial q_i}$$

where:

$$m_{ij} = \int_{mass}\left(X_iX_j + Y_iY_j + Z_iZ_j\right)dm$$

$$m_{ijk} = \int_{mass}\left(X_iX_{jk} + Y_iY_{jk} + Z_iZ_{jk}\right)dm$$

Even though Whittaker's and Olsen's expressions look simple in principle, in practice their implementation can be quite lengthy for complicated geometries with many degrees of freedom. The development of the required expressions can be greatly assisted by symbolic algebra software.

3. Example of Coupled Rigid - Flexible Motions, The "Sprung" Pendulum

In the first case we want to determine when "rigid body" motions can have important effects on the flexible motions. Consider the "sprung" pendulum which is free to rotate or oscillate about the origin in the x, y (or r, θ) plane, but which also contains a radial spring of linear stiffness k (Figure 1). We will refer to the angular motion as the "rigid body" motion and the radial motion as the "flexible" motion. Proceeding through the usual process[4] of the inertial coordinates, inertial velocities, virtual displacements, kinetic energy, potential energies (due to stiffness and gravity), the Lagrangian is:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}k\left(r - r_k\right)^2 - mg\left(r\sin\theta - Y_g\right)$$

From Lagrange's equations the radial differential equations is:

$$\boxed{\ddot{r} + \left(\omega_0^2 - \dot{\theta}^2\right)r = \frac{F_r}{m} + \frac{k}{m}r_k - g\sin\theta}$$

We also can obtain the angular equation, but we can always interpret it as the angular force required to produce the stipulated motions.

3.1 Rotation at Constant Angular Velocity

In the first case we stipulate that the pendulum moves through a complete circular motion at a constant angular velocity of ω. Then $\theta = \omega t$, and the radial differential equation is:

$$\boxed{\ddot{r} + \left(\omega_0^2 - \omega^2\right)r = \frac{F_r}{m} + \frac{k}{m}r_k - g\sin\omega t}$$

Regardless of the radial force, the radial response acts as if the natural frequency in the radial direction was

$$\omega_{effective} \rightarrow \sqrt{\omega_0^2 - \omega^2}$$

3.2 Simple Harmonic Rotation

In the second case we stipulate that the pendulum oscillates through an amplitude θ_0 with a constant frequency ω. Then $\theta = \theta_0 \sin\omega t$, and the radial differential equation is:

$$\boxed{\ddot{r} + \left[\omega_0^2 - \frac{\omega^2\theta_0^2}{2}(1+\cos 2\omega t)\right]r = \frac{F_r}{m} + \omega_0^2 r_k}$$

The complete solution of the radial equation depends on the LHS, RHS and initial conditions. The LHS can be converted, with a change of variables,

$$\tau = \omega t \quad a = \frac{\omega_0^2}{\omega^2} - \frac{1}{2}\theta_0^2 \quad b = \frac{1}{4}\theta_0^2$$

to the classical Mathieu's equation.

$$\boxed{r'' + (a - 2b\cos 2\tau)r = 0}$$

Mathieu's equation applies to the vibrations of spinning satellites, buckling of beams with periodic end forces, the saturation of loudspeakers, tides in circular bodies of water and many other problems. In our application, if the radial force does not depend on **r** then the stability of the solutions depends only on the frequency ratio $\frac{\omega_0}{\omega}$ and the angular amplitude θ_0

Intuitively, one would expect that the effects on stability would be small unless the angular amplitude is large or the frequency ratio is near 1. Figure 2 (from McLachlan[7]) shows the classical plot of the regions of stability/instability for periodic solutions of Mathieu's equation. Regions of instability are shown to be emanating from the points

$$a = \left(\frac{\omega_0}{\omega}\right)^2 - \frac{1}{2}\theta_0^2 = 1, 2^2, 3^2, \ldots n^2$$

So a question becomes - what practical values of **a, b** put the solutions into the stable or unstable regions. For instance, in the neighborhood of **a=1**, we can use McLachlan's[7] boundaries to obtain a region for instability for small **b**. Figure 3 shows the lower and upper bounds of the narrow unstable region for frequency ratios near 1 and for angular amplitudes up to 10 degrees. The instability range continues to widen for higher values of the angular amplitude. For frequency ratios near 2, 3, 4,..., the instability ranges exist, but over ever narrower ranges of angular amplitude.

We also can integrate the equation numerically. Figure 4 shows typical time histories for a frequency ratio of 1.1, damping of 0.02 and amplitudes of 0.53, 0.54 and 0.55 radians. The solutions are stable for 0.53 radians and unstable for 0.54 and 0.55 radians. Figure 5 gives a general pattern for the smallest amplitudes to produce instability for damping of 0.02 and frequency ratios up to 2.

In summary, the problem of the "sprung" pendulum shows that rigid body motions can affect the flexible motions:

a. Constant angular velocity reduces the "effective" radial natural frequency;

b. Forced sinusoidal angular motion can produce radial instability near integer values of the frequency ratio as the angular amplitudes grow large.

4. Typical section Airfoil on an Accelerating, Rotating Fuselage

The second case is a problem that is closer to practical interest - a typical section airfoil on an accelerating, rotating fuselage. We will apply Lagrange's equations in inertial coordinates, then transform the equations to noninertial, body - fixed coordinates for solution. We want to show when rigid body rotations and longitudinal accelerations must be included in the flexibility equations.

Consider a slender airfoil which is mounted on a slender fuselage. (Figure 6). The fuselage has inertial coordinates $X_0 = q_1$, $Y_0 = q_2$ and pitch angle $\theta = q_3$. The airfoil is located at fuselage position $x = x_w$ and has its own degrees of freedom in vertical translation $h = q_4$ and rotation $\delta = q_5$.

4.1 Equations of Motion

For a general point on the slender fuselage and airfoil the inertial coordinates are:

Fuselage $X = q_1 + xc_3 \quad Y = q_2 + xs_3$

Airfoil
$$X = q_1 + x_w c_3 - q_4 s_3 + \xi c_{35}$$
$$Y = q_2 + x_w s_3 + q_4 c_3 + \xi s_{35}$$

Proceeding through the usual process of the inertial coordinates, inertial velocities, virtual displacements, kinetic and potential energies (due to stiffness and gravity) and the Lagrangian for the complete fuselage/airfoil system, we establish the complete nonlinear equations in inertial coordinates. Then, it is more convenient to actually solve the problem in a body - fixed coordinate system. So we define the "apparent" body - fixed components of the vehicle's velocities and accelerations by

$$\dot{q}_x = \dot{q}_1 c_3 + \dot{q}_2 s_3 \qquad \dot{q}_y = \dot{q}_2 c_3 - \dot{q}_1 s_3$$

$$\ddot{q}_x = \ddot{q}_1 c_3 + \ddot{q}_2 s_3 \qquad \ddot{q}_y = \ddot{q}_2 c_3 - \ddot{q}_1 s_3$$

We simplify further by dropping second order terms in q_4, q_5 to obtain the linearized equations in terms of noninertial, body - fixed coordinates.

$$\begin{bmatrix} M_{11} & 0 & -\tilde{M}_{334} & 0 & -\tilde{M}_{435} \\ 0 & M_{11} & \tilde{M}_{y3} & M_{44} & \tilde{M}_{45} \\ -\tilde{M}_{334} & \tilde{M}_{y3} & \tilde{M}_{33} & \tilde{M}_{34} & \tilde{M}_{35} \\ 0 & M_{44} & \tilde{M}_{34} & M_{44} & \tilde{M}_{45} \\ -\tilde{M}_{435} & \tilde{M}_{45} & \tilde{M}_{35} & \tilde{M}_{45} & M_{55} \end{bmatrix} \begin{Bmatrix} \ddot{q}_x \\ \ddot{q}_y \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \end{Bmatrix} - \dot{q}_3 \begin{bmatrix} \tilde{M}_{y3} & 2M_{44} & 2\tilde{M}_{45} \\ \tilde{M}_{334} & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{M}_{334} & 0 & 0 \\ \tilde{M}_{335} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ k_4 & 0 \\ 0 & k_5 \end{bmatrix} \begin{Bmatrix} q_4 \\ q_5 \end{Bmatrix} + g \left(c_3 \begin{Bmatrix} 0 \\ M_{11} \\ \tilde{M}_{y3} \\ M_{44} \\ \tilde{M}_{45} \end{Bmatrix} + s_3 \begin{Bmatrix} M_{11} \\ 0 \\ -\tilde{M}_{334} \\ 0 \\ -\tilde{M}_{435} \end{Bmatrix} \right) \approx \begin{Bmatrix} Q_1 c_3 + Q_2 s_3 \\ Q_2 c_3 - Q_1 s_3 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ k_4 q_{4_{ref}} \\ k_5 q_{5_{ref}} \end{Bmatrix}$$

where

$$m_w = \int_{wing} dm$$

$$S_w = \int_{wing} \xi dm$$

$$I_w = \int_{wing} \xi^2 dm$$

$$\tilde{M}_{33} = I_f + I_w + m_w x_w^2 + 2 S_w x_w$$

$$\tilde{M}_{34} = m_w x_w + S_w$$

$$\tilde{M}_{35} = I_w + S_w x_w$$

$$\tilde{M}_{45} = S_w$$

$$\tilde{M}_{334} = m_w q_4 + S_w q_5$$

$$\tilde{M}_{335} = S_w (q_4 - x_w q_5)$$

$$\tilde{M}_{435} = S_w q_5$$

$$\tilde{M}_{y3} = S_f + x_w m_w + S_w$$

4.2 Separate Rigid and Flexible Equations - Body Coordinates

The total solution requires the solution of the five coupled equations. It can be convenient to separate the complete equations into separate rigid equations and flexible equations. If we separate the purely rigid body terms, coupling terms and purely flexible terms - the linearized flexibility equations, in body - fixed coordinates, (with damping added) are :

$$\begin{bmatrix} m_w & S_w \\ S_w & I_w \end{bmatrix} \begin{Bmatrix} \ddot{q}_4 \\ \ddot{q}_5 \end{Bmatrix} + \begin{bmatrix} c_4 & 0 \\ 0 & c_5 \end{bmatrix} \begin{Bmatrix} \dot{q}_4 \\ \dot{q}_5 \end{Bmatrix}$$

$$+ \left(\begin{bmatrix} k_4 & 0 \\ 0 & k_5 \end{bmatrix} - \dot{q}_3^2 \begin{bmatrix} m_w & S_w \\ S_w & -x_w S_w \end{bmatrix} - S_w (\ddot{q}_x + g s_3) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} q_4 \\ q_5 \end{Bmatrix}$$

$$\approx \begin{Bmatrix} Q_4 \\ Q_5 \end{Bmatrix} - \begin{bmatrix} m_w & m_w x_w + S_w \\ S_w & I_w + x_w S_w \end{bmatrix} \begin{Bmatrix} \ddot{q}_y \\ \ddot{q}_3 \end{Bmatrix} - g c_3 \begin{Bmatrix} m_w \\ S_w \end{Bmatrix} + \begin{bmatrix} k_4 & 0 \\ 0 & k_5 \end{bmatrix} \begin{Bmatrix} q_{4_{ref}} \\ q_{5_{ref}} \end{Bmatrix}$$

4.3 Vibration Solutions

Immediately we can see hints of the effects of the pitch rate $\dot{q}_3$ and the acceleration along the body axis $\ddot{q}_x$ as they alter the "effective stiffness" in the flexibility equation. Assuming $\dot{q}_3, \ddot{q}_x$ are constants, figures 7 and 8 show the effects of aircraft pitch rate on the coupled (unbalance not equal zero) airfoil frequencies of translation and rotation for the airfoil slightly aft and slight forward (1 chord) of the aircraft axis.

In the case of the unbalance equal zero, the vibration equations are uncoupled and the translation mode just acts as if the "effective" stiffness is

$$k_{4_{eff}} = k_4 - m_w \dot{q}_3^2 = k_4 \left[1 - \left(\frac{\dot{q}_3}{\omega_4} \right)^2 \right]$$

In the case where the $\dot{q}_3 = 0$ but the unbalance is not zero, the equations remain coupled, but the torsion equation acts with an "effective" stiffness

$$k_5 - S_w \ddot{q}_x = k_5 \left(1 - \frac{\hat{x}_w}{\hat{r}_w^2} \frac{g}{c \omega_5^2} \frac{\ddot{q}_x}{g} \right)$$

Whether the effective torsional stiffness is slightly larger or smaller depends on the sign of the unbalance and whether the aircraft is accelerating or decelerating.

4.4 Aeroelastic (Hypersonic) Equations

We use (for convenience) hypersonic aerodynamics from piston theory[8], to obtain the hypersonic flexibility equations.

$$\begin{bmatrix} 1 & \hat{x}_w \\ \hat{x}_w & \hat{r}_w^2 \end{bmatrix} \begin{Bmatrix} \hat{q}_4'' \\ q_5'' \end{Bmatrix} + \left(2 \begin{bmatrix} \lambda_4 \zeta_4 & 0 \\ 0 & \hat{r}_w^2 \lambda_5 \zeta_5 \end{bmatrix} + \frac{4\mu}{M_\infty} \begin{bmatrix} 1 & \hat{F}_1 \\ \hat{F}_1 & \hat{F}_2 \end{bmatrix} \right) \begin{Bmatrix} \hat{q}_4' \\ q_5' \end{Bmatrix}$$

$$+ \left(\begin{bmatrix} \lambda_4^2 & 0 \\ 0 & \hat{r}_w^2 \lambda_5^2 \end{bmatrix} - \frac{4\mu}{M_\infty} \begin{bmatrix} 0 & 1 \\ 0 & \hat{F}_1 \end{bmatrix} - q_3'^2 \begin{bmatrix} 1 & \hat{x}_w \\ \hat{x}_w & -\overset{*}{x}_w \hat{x}_w \end{bmatrix} - \hat{x}_w (\hat{q}_x'' + \hat{g} s_3) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \hat{q}_4 \\ q_5 \end{Bmatrix}$$

$$\approx \frac{4\mu}{M_\infty} \begin{bmatrix} 1 & -(\overset{*}{x}_w + \hat{F}_1) \\ \hat{F}_1 & -(\overset{*}{x}_w \hat{F}_1 + \hat{F}_2) \end{bmatrix} \begin{Bmatrix} \alpha \\ q_3' \end{Bmatrix} - \begin{bmatrix} 1 & \overset{*}{x}_w + \hat{x}_w \\ \hat{x}_w & \hat{r}_w^2 + \overset{*}{x}_w \hat{x}_w \end{bmatrix} \begin{Bmatrix} \hat{q}_y'' \\ q_3'' \end{Bmatrix} - \hat{g} c_3 \begin{Bmatrix} 1 \\ \hat{x}_w \end{Bmatrix} + \begin{bmatrix} \lambda_4^2 & 0 \\ 0 & \hat{r}_w^2 \lambda_5^2 \end{bmatrix} \begin{Bmatrix} \hat{q}_{4_{ref}} \\ q_{5_{ref}} \end{Bmatrix}$$

where the nondimensional variables are defined by:

lengths

c=chord l=span

positions/coordinates $x_w = c\overset{*}{x}_w \quad \xi_{le} = c\hat{\xi}_{le}$

$$q_4 = c\hat{q}_4 \quad q_x = c\hat{q}_x \quad q_y = c\hat{q}_y$$

inertias

$$S_w = m_w c\hat{x}_w \quad I_w = m_w c^2 \hat{r}_w^2$$

stiffnesses

$$k_4 = m_w \omega_4^2 \quad k_5 = I_w \omega_5^2 = m_w c^2 \hat{r}_w^2 \omega_5^2$$

dampings

$$c_4 = 2m_w \omega_4 \varsigma_4 \quad c_5 = 2m_w c^2 \hat{r}_w^2 \omega_5 \varsigma_5$$

time and frequency

$$t = \tau \, c/V_\infty \quad \lambda = \frac{c\omega}{V_\infty}$$

time derivative

$$\dot{f}(t) = f'(\tau)\frac{V_\infty}{c} \quad \ddot{f}(t) = f''(\tau)\left(\frac{V_\infty}{c}\right)^2$$

air density and gravity

$$\mu = \frac{\rho_\infty l c^2}{2m_w} \quad \hat{g} = \frac{gc}{V_\infty^2}$$

aerodynamic coefficients

$$Q_4 = N_w = \frac{\rho V_\infty^2 lc}{2} C_{N_w} \quad Q_5 = M_w = \frac{\rho V_\infty^2 lc^2}{2} C_{M_w}$$

Geometry integrals

$$\hat{F}_1 = \hat{\xi}_{le} - 0.5 \quad \hat{F}_2 = \hat{\xi}_{le}^2 - \hat{\xi}_{le} + 0.333...$$

As usual, the damping is modified with the aerodynamic damping and the stiffness is modified with the aerodynamic stiffness. However, the stiffness also has terms that are proportional to the nondimensional pitch rate q_3' and the nondimensional acceleration q_x''.

4.5 <u>Rigid Body Motions as Constant Parameters in the Aeroelastic Equations</u>

Now again consider the pitch rate and the aircraft acceleration as constant parameters. From the differential equation and the vibration solutions, we know that the rigid body pitch rate <u>will</u> decrease the bending frequency (even if only slightly), and that the rigid body acceleration (or deceleration) along the body axis <u>can</u> increase or decrease the torsional frequency.

Therefore those rigid body motions also can alter the aircraft speeds for aeroelastic divergence and flutter. For example, under the assumed conditions we can obtain an approximate expression for the μ required for aeroelastic divergence:

$$\mu_{Div} = \frac{M_\infty}{4}\frac{\lambda_5^2 \hat{r}_w^2}{\hat{F}_1}\left[1 - \varepsilon^2 \frac{\hat{x}_w}{\hat{r}_w}\left(\frac{\hat{r}_w}{\hat{F}_1} - \frac{\overset{*}{x}_w}{\hat{r}_w}\phi^2\right) - \frac{\hat{x}_w}{\hat{r}_w}\left(\frac{\hat{q}_x'' + \hat{g}s_3}{\lambda_5^2 \hat{r}_w}\right)\right]$$

where

$$\varepsilon = q_3'/\lambda_4 \quad \phi = \lambda_4/\lambda_5$$

which shows the importance on the divergence speed of:

a. ε, the ratio of the pitch rate to the uncoupled translation frequency;

b. $\frac{\hat{q}_x''}{\lambda_5^2 \hat{r}_w}$, the relationship of the acceleration to the torsional frequency and the radius of gyration

c. ϕ, the ratio of the uncoupled translation and rotation frequencies

d. $\frac{\hat{x}_w}{\hat{r}_w}$, the relationship of the unbalance to the radius of gyration.

We also can use the hypersonic aeroelastic equation to do an eigenvalue calculation (dropping the RHS) to obtain flutter solutions. Figures 9 and 10 show representative effects of pitch rate and acceleration on hypersonic divergence and flutter boundaries.

4.6 <u>Forced Rigid Body Motions in the Aeroelastic Equations</u>

Rather than assume that the rigid body motions are constant parameters, we can assume representative forms for their time dependent motions and then plug them into the flexible equations of motion. We need the terms $\alpha, \dot{q}_3, \dot{q}_3^2, \ddot{q}_x, \ddot{q}_y$ and $\ddot{q}_3$. Following Etkin's[9] notation we can <u>assume</u> the time dependent forms for the oscillatory, damped speed, pitch angle and angle of attack, wherein each expression the terms **a** and **b** are assumed constants:

$$\dot{q}_x = u_0 + E(a_u S_1 + b_u C_1)$$

$$\theta = q_3 = \theta_0 + E(a_\theta S_1 + b_\theta C_1)$$

$$\alpha = \alpha_0 + E(a_\alpha S_1 + b_\alpha C_1)$$

where

$$S_n = \sin n\omega_{rb}t \quad C_n = \cos n\omega_{rb}t \quad E = e^{-\omega_{rb}\varsigma_{rb}t}$$

$$\omega_{rb} = \text{assumed "rigid body" frequency}$$

$$\varsigma_{rb} = \text{assumed "rigid body" damping}$$

Noting that $\dot{q}_y \approx \dot{q}_x \alpha$ and combining angles where possible, we obtain the "forcing terms" to be included in the equations of motion. The terms $\dot{q}_3^2$, $\ddot{q}_x$ appear on the LHS of the equation and influence stability:

$$\ddot{q}_x = \omega_{rb} E\left(a_{\ddot{q}_x} S_1 + b_{\ddot{q}_x} C_1\right)$$

$$\dot{q}_3^2 = \tfrac{1}{2}\omega_{rb}^2 E^2\left(K_{\dot{q}_3^2} + \overset{*}{a}_{\dot{q}_3^2} S_2 + \overset{*}{b}_{\dot{q}_3^2} C_2\right)$$

The major point is that terms $S_2 = \sin 2\omega_{rb}t$ and $C_2 = \cos 2\omega_{rb}t$ on the LHS introduce behavior like that of Mathieu's equation.

5. The General Case - Three Dimensional Motion of a Flexible Vehicle

5.1 Geometry

We start with an inertial X, Y, Z coordinate system and a noninertial x, y, z system that can accelerate and rotate in the X,Y,Z system (Figure 11). The origin of the x, y, z is located in the inertial system at:

$$X = X_0 = q_1 \quad Y = Y_0 = q_2 \quad Z = Z_0 = q_3$$

The orientation of the x, y, z system is given by the conventional sequence of Euler rotations:

$$\psi = q_4 \quad \theta = q_5 \quad \phi = q_6$$

5.2 Inertial Coordinates

Then inertial coordinates of a general point in x, y, z are:

$$\{X,Y,Z\} = \{q_1,q_2,q_3\} + [\tau]\{x,y,z\}$$

where $[\tau]$, the Euler transformation[9], is the product of three transformations that depend on the Euler angles:

$$[\tau] = [\tau_4][\tau_5][\tau_6]$$

$$[\tau_4] = \begin{bmatrix} c_4 & -s_4 & 0 \\ s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\tau_5] = \begin{bmatrix} c_5 & 0 & s_5 \\ 0 & 1 & 0 \\ -s_5 & 0 & c_5 \end{bmatrix} \quad [\tau_6] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_6 & -s_6 \\ 0 & s_6 & c_6 \end{bmatrix}$$

$$s_i = \sin q_i \quad c = \cos q_i$$

We also write the "local" coordinates in terms of additional generalized coordinates $q_7, q_8 \ldots q_n$:

$$\{x, y, z\} = \sum_{i=7}^{n} \{x_i, y_i, z_i\} q_i(t)$$

to obtain the inertial coordinates in terms of the generalized coordinates:

$$\{X, Y, Z\} = \{q_1, q_2, q_3\} + [\tau] \sum_{i=7}^{n} \{x_i, y_i, z_i\} q_i(t)$$

The kinetic energy, in terms of the inertial coordinates is a lengthy expression which shows why it can be useful to use Olsen's[6] form of the equations of motion (which doesn't require the kinetic energy), rather than Whittaker's[5] form which does require the kinetic energy.

5.3. Overcoming the Tedious Aspects - Quasi Coordinates

Again, we could use Lagrange's equations on the Lagrangian in inertial coordinates to obtain the equations of motion for the flexible system. We would be accurately accounting for all of the inertia couplings that arise from the fact that the noninertial x, y, z system is accelerating and rotating in the inertial X, Y, Z system. We could solve the problem in terms of the inertial translations q_1, q_2, q_3 and the Euler angles q_4, q_5, q_6 and then transform the results to the translations along the body axes q_x, q_y, q_z and the instantaneous angular velocities $\omega_x, \omega_y, \omega_z$, using the transformations:

$$\{q_x, q_y, q_z\} = [\tau]^T \{q_1, q_2, q_3\}$$

$$\{\omega_x, \omega_y, \omega_z\} = [\alpha]\{\dot{q}_4, \dot{q}_5, \dot{q}_6\}$$

where

$$[\alpha] = \begin{bmatrix} -s_5 & 0 & 1 \\ c_5 s_6 & c_6 & 0 \\ c_5 c_6 & -s_6 & 0 \end{bmatrix}$$

The approach is correct in principal. However, it works easiest for special cases like rotation about one axis (where the time derivative of the appropriate Euler angle is indeed the angular velocity). However, it suffers from two shortcomings in the general case of three dimensional motions.

First, the generalized coordinates q_1, q_2, q_3 are the translations in the directions of the inertial coordinates. We would like to replace them with the translations in directions of the noninertial, body - axis coordinates q_x, q_y, q_z.

Second, the generalized coordinates q_4, q_5, q_5 are the Euler angles. Their time derivatives $\dot{q}_4, \dot{q}_5, \dot{q}_5$ may not be the physical angular velocities of the x, y, z system for general motions. We would like to replace them with the physical angular velocities of the noninertial, body - axis coordinates, $\omega_x, \omega_y, \omega_z$.

However, a much more elegant and simple method is available, the method of quasi - coordinates due to Hamel[10] and Boltzmann[11]. The term "quasi - coordinates" refers to the fact that we cannot (in the general case of three dimensional motions) directly integrate the angular velocities to get the generalized coordinates. Actually all we are doing is performing the transformations before we apply Lagrange's equations to obtain the differential equations, rather than after we get them.

Whittaker[5] and Meirovitch[12] explain the method of quasi - coordinates for the special case of rotational motions. Several others, among them Nayfeh and Mook[13], give applications.

The basic idea is that we want to write Lagrange's equations in a form that treats directly the body axis translations q_x, q_y, q_z and the true angular velocities $\omega_x, \omega_y, \omega_z$. We start with the usual form of Lagrange's equations in terms of the original, independent generalized coordinates $q_1, q_2 \ldots q_n$ and their time derivatives $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = Q_i$$

The Lagrangian can be written in the usual form in the original inertial coordinates:

$$L = L\begin{pmatrix} q_1, q_2, q_3, q_4, q_5, q_6, q_7, \cdots q_n; \\ \dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \dot{q}_5, \dot{q}_6, \dot{q}_7, \cdots \dot{q}_n \end{pmatrix}$$

If we note that:

$$\{\dot{q}_4, \dot{q}_5, \dot{q}_6\} = [\beta]\{\omega_x, \omega_y, \omega_z\}$$

where

$$[\beta] = [\alpha]^{-1} = \begin{bmatrix} 0 & c_5^{-1} s_6 & c_5^{-1} c_6 \\ 0 & c_6 & -s_6 \\ 1 & t_5 s_6 & t_5 c_6 \end{bmatrix}$$

we can obtain the equivalent form of the Lagrangian in the quasi - coordinates:

$$\tilde{L} = \tilde{L}\begin{pmatrix} q_x, q_y, q_z, q_4, q_5, q_6, q_7, \cdots q_n; \\ \dot{q}_x, \dot{q}_y, \dot{q}_z, \omega_x, \omega_y, \omega_z, \dot{q}_7, \cdots \dot{q}_n \end{pmatrix}$$

Then, following Whittaker[5] we can obtain the equations for:

Translation DOFs:

$$\boxed{\frac{d}{dt}\begin{Bmatrix} \frac{\partial \tilde{L}}{\partial \dot{q}_x} \\ \frac{\partial \tilde{L}}{\partial \dot{q}_y} \\ \frac{\partial \tilde{L}}{\partial \dot{q}_z} \end{Bmatrix} - \begin{Bmatrix} \frac{\partial \tilde{L}}{\partial q_x} \\ \frac{\partial \tilde{L}}{\partial q_y} \\ \frac{\partial \tilde{L}}{\partial q_z} \end{Bmatrix} + [\Omega]\begin{Bmatrix} \frac{\partial \tilde{L}}{\partial \dot{q}_x} \\ \frac{\partial \tilde{L}}{\partial \dot{q}_y} \\ \frac{\partial \tilde{L}}{\partial \dot{q}_z} \end{Bmatrix} = [\tau]^T \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}}$$

Rotation DOFs:

$$\boxed{\frac{d}{dt}\begin{Bmatrix} \frac{\partial \tilde{L}}{\partial \omega_x} \\ \frac{\partial \tilde{L}}{\partial \omega_y} \\ \frac{\partial \tilde{L}}{\partial \omega_z} \end{Bmatrix} - [\beta]^T \begin{Bmatrix} \frac{\partial \tilde{L}}{\partial q_4} \\ \frac{\partial \tilde{L}}{\partial q_5} \\ \frac{\partial \tilde{L}}{\partial q_6} \end{Bmatrix} + [\Omega]\begin{Bmatrix} \frac{\partial \tilde{L}}{\partial \omega_x} \\ \frac{\partial \tilde{L}}{\partial \omega_y} \\ \frac{\partial \tilde{L}}{\partial \omega_z} \end{Bmatrix} = [\beta]^T \begin{Bmatrix} Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix}}$$

Flexible DOFs remain the same (except that we must use the modified Lagrangian $\tilde{L}$)

$$\boxed{\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}_i} - \frac{\partial \tilde{L}}{\partial q_i} = Q_i \quad for\ i \geq 7}$$

where:

$$[\Omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

These are the equations of motion in terms of quasi - coordinates. They are the fundamental advance which allows us to formulate a unified set of equations that can be used without simplification for the aerodynamics, structures and stability and control of flexible aircraft - they allow us to place the aeroelastic problem into a coordinate system and notation that is used in flight mechanics and stability and control.

5.4 Energies - Noninertial Body - Axis Coordinates

The kinetic energy in terms of the body - axis variables is:

$$\begin{aligned} KE = {} & \tfrac{1}{2}M\left(\dot{q}_x^2 + \dot{q}_y^2 + \dot{q}_z^2\right) \\ & + S_x\left(\dot{q}_y\omega_z - \dot{q}_z\omega_y\right) + S_y\left(\dot{q}_z\omega_x - \dot{q}_x\omega_z\right) + S_z\left(\dot{q}_x\omega_y - \dot{q}_y\omega_x\right) \\ & + \tfrac{1}{2}I_{xx}\left(\omega_y^2 + \omega_z^2\right) - I_{xy}\omega_x\omega_y - I_{xz}\omega_x\omega_z \\ & + \tfrac{1}{2}I_{yy}\left(\omega_x^2 + \omega_z^2\right) - I_{yz}\omega_y\omega_z + \tfrac{1}{2}I_{zz}\left(\omega_x^2 + \omega_y^2\right) \\ & + S_{\dot{x}}\dot{q}_x + S_{\dot{y}}\dot{q}_y + S_{\dot{z}}\dot{q}_z \\ & + \left(I_{x\dot{y}} - I_{\dot{x}y}\right)\omega_z + \left(I_{y\dot{z}} - I_{\dot{y}z}\right)\omega_x + \left(I_{z\dot{x}} - I_{\dot{z}x}\right)\omega_y \\ & + \tfrac{1}{2}\left(I_{\dot{x}\dot{x}} + I_{\dot{y}\dot{y}} + I_{\dot{z}\dot{z}}\right) \end{aligned}$$

where typical inertia integrals are:

$$M = \int_{body} dm$$

$$S_x = \int x\,dm = \sum_i \left(\int x_i\,dm\right) q_i = \sum_i S_{x_i} q_i$$

$$S_{\dot{x}} = \int \dot{x}\,dm = \sum_i \left(\int x_i\,dm\right) \dot{q}_i = \sum_i S_{x_i} \dot{q}_i$$

$$I_{xy} = \int_{body} xy\,dm = \sum_{i,j} \left(\int x_i y_j\,dm\right) q_i q_j = \sum_{i,j} I_{x_i y_j} q_i q_j$$

$$I_{\dot{x}y} = \int_{body} \dot{x}y\,dm = \sum_{i,j} \left(\int x_i y_j\,dm\right) \dot{q}_i q_j = \sum_{i,j} I_{x_i y_j} \dot{q}_i q_j$$

The potential energy due to gravity will come from our gravitational model. In the case of a "flat earth":

$$V_g = -g \int_{mass} \left(Z - Z_{ref}\right) dm$$

$$= -g\left(q_3 - Z_{ref}\right)m - g\lfloor 0 \quad 0 \quad 1 \rfloor [\tau] \sum_{i=7}^{n} \begin{Bmatrix} S_{x_i} \\ S_{y_i} \\ S_{z_i} \end{Bmatrix} q_i$$

We expect the potential energy due to flexibility to be of the form

$$V_f = \tfrac{1}{2} \sum_{i=7}^{n} \sum_{j=7}^{n} V_{b_{ij}} q_i q_j$$

Or perhaps a more general expression for larger deflections

$$V_f = \tfrac{1}{2} \sum_{i=7}^{n} \sum_{j=7}^{n} V_{b_{ij}} q_i q_j + \tfrac{1}{2} \sum_{i=7}^{n} \sum_{j=7}^{n} \sum_{k=7}^{n} V_{b_{ijk}} q_i q_j q_j$$

The author has performed those operations, and the complete set of differential equations is available (but too lengthy to present here).

5.6 Simplification to the "Flat" Airplane

If we specialize the general body to consider an essentially "flat" surface in the xy plane (Right handed xyz coordinate system attached to the body), Figure 12, and make the usual definitions

$$\dot{q}_x = U \quad \text{velocity along the x axis}$$

$$\dot{q}_x = V \quad \text{velocity along the y axis}$$

$$\dot{q}_x = W \quad \text{velocity along the z axis}$$

$$\omega_x = P \quad \text{angular velocity about the x axis}$$

$$\omega_y = Q \quad \text{angular velocity about the y axis}$$

$$\omega_z = R \quad \text{angular velocity about the z axis}$$

$$z(x,y,t) = \sum_i z_i(x,y) q_i(t)$$

The equations of motion become

Translation

$$M\begin{Bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & -S_y \\ 0 & 0 & S_x \\ S_y & -S_x & 0 \end{bmatrix} \begin{Bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{Bmatrix} + \begin{Bmatrix} WQ - VR \\ UR - WP \\ VP - UQ \end{Bmatrix} M + \begin{bmatrix} -\left(Q^2 + R^2\right) & PQ \\ PQ & -\left(P^2 + R^2\right) \\ PR & QR \end{bmatrix} \begin{Bmatrix} S_x \\ S_y \end{Bmatrix}$$

$$+ g\begin{Bmatrix} s_5 \\ -c_5 s_6 \\ -c_5 c_6 \end{Bmatrix} M + \sum_i \left(\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \ddot{q} + 2\begin{Bmatrix} Q \\ -P \\ 0 \end{Bmatrix} \dot{q}_i + q_i \begin{Bmatrix} \left(\dot{Q} + PR\right) \\ -\left(\dot{P} - QR\right) \\ -\left(P^2 + Q^2\right) \end{Bmatrix} \right) S_{z_i} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

Rotation

$$\begin{bmatrix} 0 & 0 & S_y \\ 0 & 0 & -S_x \\ -S_y & S_x & 0 \end{bmatrix} \begin{Bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{Bmatrix} + \begin{bmatrix} I_y & -I_{xy} & 0 \\ -I_{xy} & I_x & 0 \\ 0 & 0 & I_x + I_y \end{bmatrix} \begin{Bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & PR & QR \\ -PR & -QR & 0 \\ PQ & Q^2 - P^2 & -PQ \end{bmatrix} \begin{Bmatrix} I_x \\ I_y \\ I_{xy} \end{Bmatrix} + \begin{bmatrix} VQ + WR & -UQ \\ VP & UP + WR \\ -WP & -WQ \end{bmatrix} \begin{Bmatrix} S_x \\ S_y \end{Bmatrix} + g\begin{bmatrix} 0 & -c_5 c_6 \\ c_5 c_6 & 0 \\ -c_5 s_6 & -s_5 \end{bmatrix} \begin{Bmatrix} S_x \\ S_y \end{Bmatrix}$$

$$+ \sum_i \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{q}_i + \begin{bmatrix} -V & 0 & 0 \\ U & 0 & 0 \\ 0 & -2P & -2Q \end{bmatrix} \dot{q}_i + \begin{bmatrix} -\left(\dot{V} + UR\right) & -\left(\dot{R} + PQ\right) & \left(R^2 - Q^2\right) \\ \left(\dot{U} - VR\right) & \left(P^2 - R^2\right) & -\left(\dot{R} - PQ\right) \\ \left(UP + VQ\right) & -\left(\dot{P} - QR\right) & -\left(\dot{Q} + PR\right) \end{bmatrix} q_i \right) \begin{Bmatrix} S_z \\ S_{xz} \\ S_{yz} \end{Bmatrix}$$

$$+ g\begin{bmatrix} c_5 s_6 & 0 & 0 \\ s_5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sum_i \begin{Bmatrix} S_z \\ S_{xz} \\ S_{yz} \end{Bmatrix} q_i + \sum_{i,j} \begin{Bmatrix} 2\dot{q}_j P + q_j\left(\dot{P} - QR\right) \\ 2\dot{q}_j Q + q_j\left(\dot{Q} + PR\right) \\ 0 \end{Bmatrix} I_{z_i z_j} q_i = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}$$

Flexible

$$\left(\dot{W} + VP - UQ\right)S_{z_i} - \left(\dot{Q} - PR\right)S_{xz_i} + \left(\dot{P} + QR\right)S_{y_i}$$

$$+ \sum_j \left[\ddot{q}_j - \left(P^2 + Q^2\right) q_j \right] I_{z_i z_j} + \frac{\partial V_e}{\partial q_i} - g c_5 c_6 S_{z_i} = Q_i$$

The equations above are the equations to solve for the static and dynamic response and stability of a flexible, "flat" aircraft under steady state flight or in accelerations and maneuvers. They are nonlinear and mutually couple the overall rigid body motions with the flexible deflections. They can be used for analyses of aircraft performance, stability and control, flight loads, control effectiveness and aeroelastic divergence and flutter. Of course they are more complicated than the conventional nonlinear equations for rigid body motions or the linear equations for aeroelastic response (which are coupled to the rigid body equations only through the aerodynamics).

5.7 Perfect Masses and Modes

In the special case of

a. Mass symmetry about the y axis

b. Origin at the center of mass
c. Perfectly orthogonal free - free modes

the equations of motion simplify further to the usual rigid body equations and the modified flexible equations:

Translation

$$\begin{Bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{Bmatrix} + \begin{Bmatrix} WQ - VR \\ UR - WP \\ VP - UQ \end{Bmatrix} + g \begin{Bmatrix} s_5 \\ -c_5 s_6 \\ -c_5 c_6 \end{Bmatrix} = \frac{1}{M} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

Rotation

$$\begin{Bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{Bmatrix} + \begin{Bmatrix} QR \\ -PR \\ \frac{I_{xx} - I_{yy}}{I_{xx} + I_{yy}} PQ \end{Bmatrix} = \begin{Bmatrix} M_x / I_{yy} \\ M_y / I_{xx} \\ M_z / \left(I_{xx} + I_{yy}\right) \end{Bmatrix}$$

Flexible

$$\left\{ \ddot{q}_i + \left[\omega_i^2 - \left(P^2 + Q^2\right) \right] q_i \right\} I_{z_i z_i} = Q_i$$

where

ω_i = natural frequency of the perfect ith mode

If we assume, for the moment, that the angular velocities **P** and **Q** are constant, then one approximation would be to treat the flexible equations as if the effective structural frequency for any mode is just replaced by

$$\omega_i^2 \rightarrow \omega_i^2 - \left(P^2 + Q^2\right)$$

On the other hand, since **P** and **Q** will be functions of time, the actual behavior will be more like the behavior of solutions to Mathieu's equations.

Many recent developments numerically integrate the linearized equations of motion with nonlinear aerodynamics on the RHS. It seems that, once the analyst has committed to numerical integration of the equations of motion, there is very little additional labor (or computational time) to use the more comprehensive equations of motion above.

6. Summary, Conclusions

1. Whittaker's and Olsen's expressions can be useful to formulate the equations of motion for complicated geometries with many degrees of freedom.

2. The "sprung pendulum" shows that a rigid body motion with constant angular velocity can reduce the "effective" natural frequencies. Using the resemblance to Mathieu's equation, we have seen that there is a coupling mechanism between the rigid body and flexible motions, even in the absence of aerodynamics. It appears that if a flexible frequency is up to 1.3 - 1.5 times a rigid body frequency, then those coupling effects should be considered. In some cases ("slender" aircraft) the natural frequencies may already be in those ranges. In other cases (the X - 29) the aerodynamic forces drive some of the flexible frequencies down toward the rigid body frequencies.

3. The airfoil on an accelerating/rotating fuselage shows that the effective bending stiffness is reduced by a constant pitch rate. It also shows that torsional stiffness is increased or decreased by constant acceleration/deceleration, depending on the sign of the unbalance. The results modify the divergence and flutter speeds. If we impose the rigid body motions as forced, sinusoidal, damped motions - then terms appear in the differential equations which can produce additional instabilities, such as in Mathieu's equation.

4. In the case of general motion of a flexible body, the combination of energy methods and quasi - coordinates can produce a practical set of equations that govern the aerodynamics, flight mechanics and structures problems of flexible aircraft. They allow the determination of the effects of structural flexibility on aircraft performance, stability and air loads and the effects of the "rigid body" motions on aeroelastic control - effectiveness, divergence and flutter.

5. For the special case of the "flat airplane" with perfect mass distribution and perfect modes, a simple preliminary estimate of the effects of rigid body motions on flexible motions would be to replace all of the structural frequencies by:

$$\omega_i^2 \rightarrow \omega_i^2 - \left(P^2 + Q^2\right)$$

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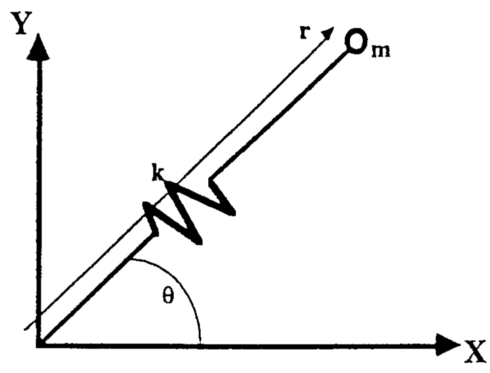

Figure 1. The Sprung Pendulum

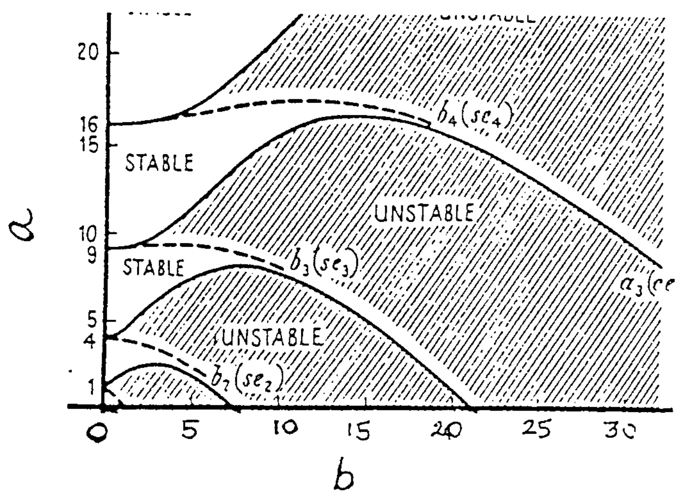

Figure 2. Regions of Stability/Instability for Periodic Solutions of Mathieu's Equation

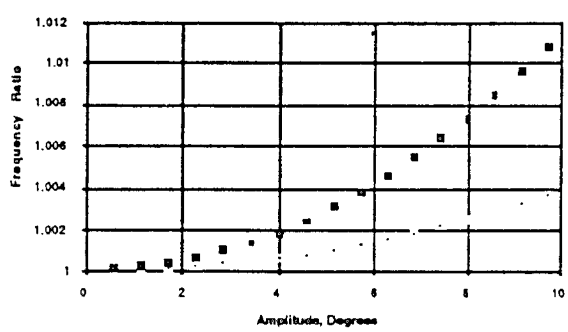

Figure 3. Instability Region for Frequency Ratios near 1

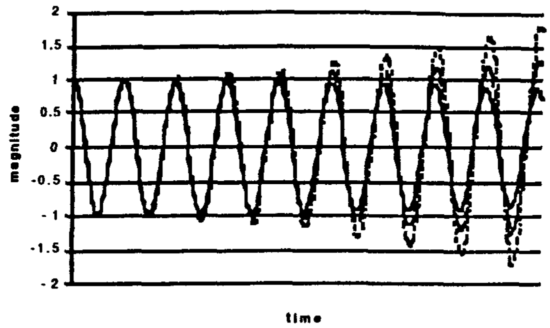

Figure 4. Time Histories for Frequency ratio 1.1

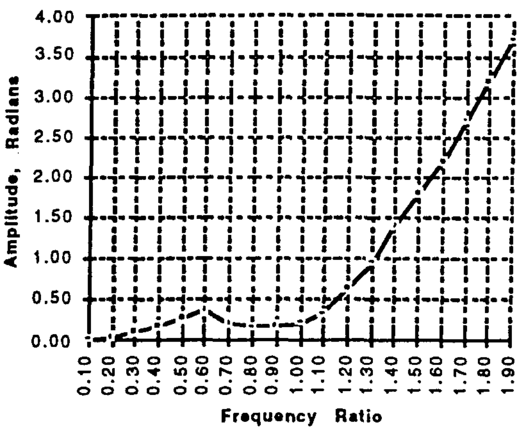

Figure 5. Smallest value of Amplitude to Become Unstable

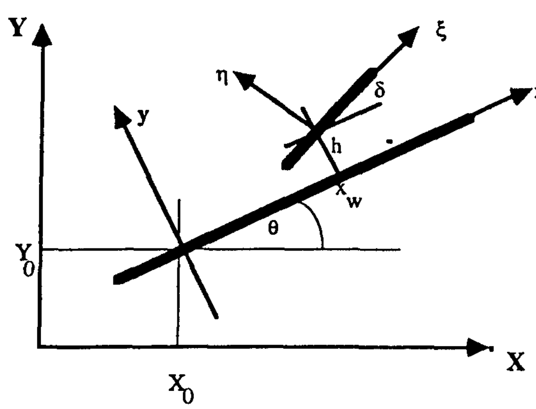

Figure 6. Airfoil on an Accelerating, Rotating Fuselage

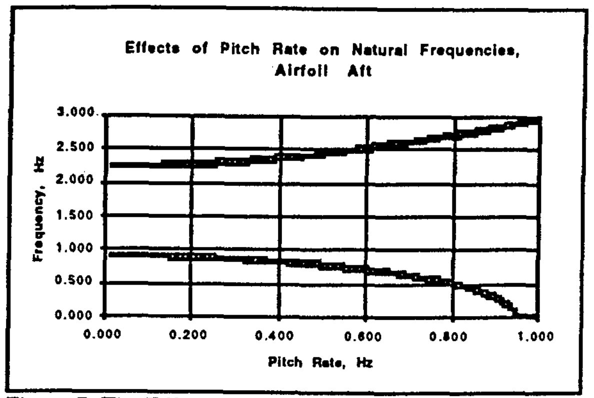

Figure 7. The Effects of pitch Rate on Natural Frequencies, $x_w^* = -1.0$

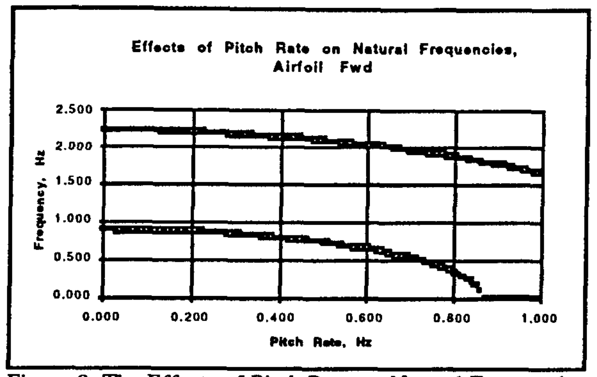

Figure 8. The Effects of Pitch Rate on Natural Frequencies,

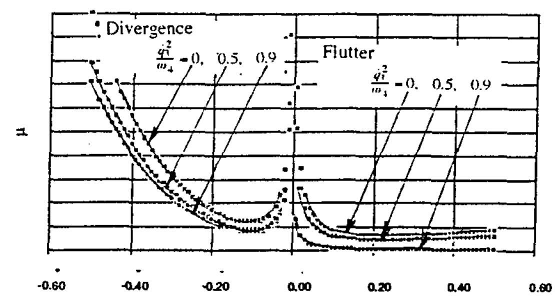

Figure 9. The Effects of Pitch Rate on Hypersonic Divergence and Flutter

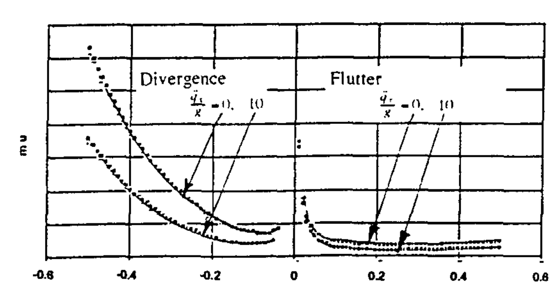

Figure 10. The Effects of Acceleration on Hypersonic Divergence and Flutter

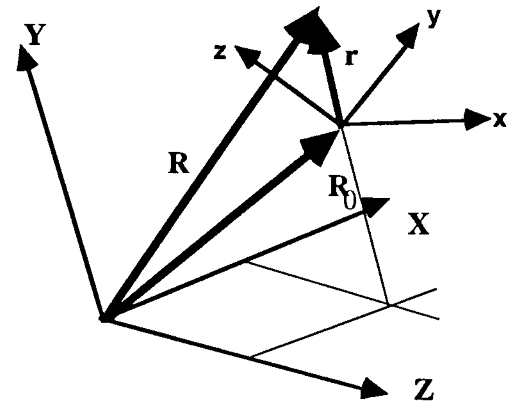

Figure 11. General Motion of a Flexible Vehicle

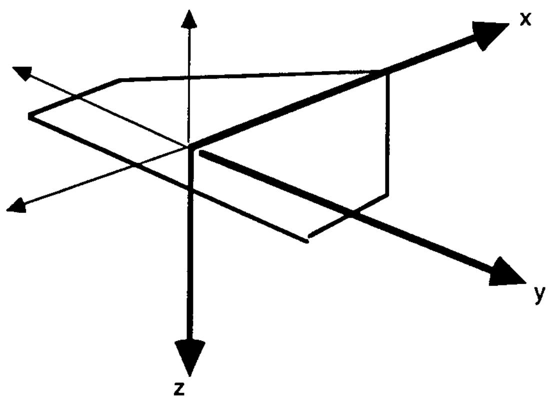

Figure 12. The "Flat"Airplane